APPROXIMATE ANALYSIS OF NATURAL CONVECTION IN A FLAT CHANNEL WITH STABILIZED FLUID FLOW

G. N. Dul'nev and A. I. Kaidanov

UDC 536.25

The boundary-layer equation is solved in approximation for stabilized flow in a channel. Formulas are derived for the local and mean Nusselt numbers, and the limits of their applicability are indicated. The results are compared with the data of other investigators.

There are numerous studies dealing with heat exchange in flat and cylindrical channels. Solution methods and research results for forced convection have been adequately discussed in [1]. Data on natural convection can be found in individual papers [2-7].

One feature of the natural-convection problem is the complexity involved in specifying the conditions at the channel inlet. In contrast to forced-convection problems, where the formulations are correct, investigations of natural convection have made use of various models that have not received sufficient validation and which, as we shall show, are nearly equivalent.

In almost all the natural-convection studies known to us, a channel with isothermal walls is considered; only [7] reports experimental results for walls with constant losses to the flowing fluid.

Finally, again in contrast to forced-convection problems, for the most part both the analytic and experimental studies give the mean, rather than the local Nusselt number.

We shall consider natural convection in a flat vertical channel heated symmetrically, with an arbitrary distribution of wall temperature over the height (Fig. 1).

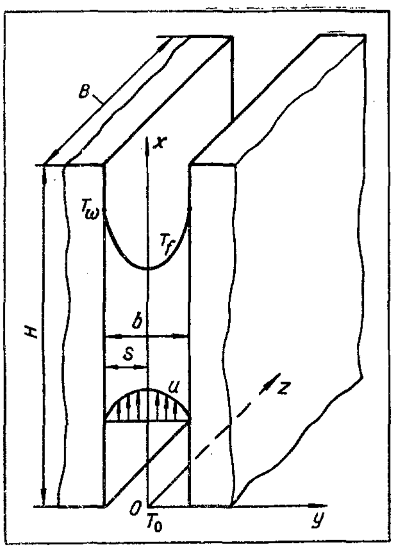

Fig. 1. Configuration of channel and basic dimensions.

In formulating the problem, we make the usual assumptions ([1], p. 76, Assumptions 1, 2, 4, 6), while introducing the following restrictions: a) the channel temperature field is symmetric about the plane y = 0; b) the wall temperature depends solely on the longitudinal coordinate: $T_W = T_W(x)$; c) the width of the channel is considerably less than the depth ($b \ll B$); and d) the width is much less than the height ($b \ll H$).

By virtue of restriction c), the plane boundary layer equations are valid for the fluid in the channel [4]:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu_f\frac{\partial^2 u}{\partial y^2}+\beta g(T_f-T_0); \tag{1}$$

$$u\frac{\partial T_f}{\partial x}+v\frac{\partial T_f}{\partial y}=a_f\frac{\partial^2 T_f}{\partial y^2}; \tag{2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{3}$$

The boundary conditions are written as

$$y=\pm S,\ x>0,\ u=0,\ v=0; \tag{4}$$

$$y=\pm S,\ x>0,\ T_f=T_w(x); \tag{5}$$

Institute of Precision Mechanics and Optics, Leningrad. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 17, No. 2, pp. 216-255, August, 1969. Original article submitted October 21, 1968.

$$x \leqslant 0,\ u = 0,\ v = 0; \tag{6}$$

$$x \leqslant 0,\ T_f = T_0. \tag{7}$$

We assume that sufficiently far from the inlet ($x \geq l_{ih}$), the velocity profile is parabolic within the channel, and does not depend on x. By virtue of restriction d), the length of the initial hydrodynamic section is small as compared with the overall channel length, so that we can take $l_{ih} \approx 0$ in approximation. This assumption, which is strictly true only for a stabilized flow [1], can be written in the form

$$u = u_{ax}(1 - \bar{y}^2) \text{ for } \quad x > 0. \tag{8}$$

We find u_{ax} from (1), multiplying it by ρ_f and integrating over the channel volume. Integrating with respect to y, from (3) and (4) we have

$$\int\limits_{-S}^{+S} v \frac{\partial u}{\partial y}\, dy = uv\Big|_{-S}^{+S} - \int\limits_{-S}^{+S} u \frac{\partial v}{\partial y} = \int\limits_{-S}^{+S} u \frac{\partial u}{\partial x}\, dy.$$

We now integrate with respect to x, z and use the expression (8) for the velocity:

$$R + A = \Delta I; \tag{9}$$

$$A = 2SBHg\rho_f (T_{fv} - T_0);\ T_{fv} = \frac{\int\limits_{-S}^{+S}\int\limits_{0}^{B}\int\limits_{0}^{H} T_f dydzdx}{2SBH}; \tag{10a}$$

$$\Delta I = 2\rho_f BS \left[u_{ax}^2 \int\limits_0^1 (1 - \bar{y}^2)^2\, d\bar{y} - 0 \right] = \frac{16}{15}\, \rho_f BSu_{ax}^2; \tag{10b}$$

$$R = 2BH\mu_f \frac{\partial u}{\partial y}\Big|_0^S = -\frac{4\mu_f u_{oc} BH}{S}. \tag{10c}$$

Relationship (9) has a simple physical interpretation: the algebraic sum of the friction force and Archimedes force equals the change in momentum.

We let

$$M = \mathrm{Gr}\,\mathrm{Pr}\frac{b}{H};\ \mathrm{Gr} = \frac{\beta g(\bar{T}_w - T_0)\, b^3}{\nu_f^2};\ \bar{T}_w = \frac{\int\limits_0^H T_w(x)\, dx}{H};$$

$$\psi = \frac{T_{fv} - T_0}{\bar{T}_w - T_0};\ \mathrm{Pr} = \frac{\nu_f}{a_f};\ \mathrm{Pe}^* = \mathrm{Pe}^* \frac{b}{H}; \tag{11}$$

$$\mathrm{Pe} = \frac{u_a b}{a_f};\ u_a = \int\limits_0^1 u(\bar{y})\, d\bar{y} = \frac{2}{3}\, u_{ax}.$$

Solving (9) for u_{ax}, we obtain

$$u_{ax} = \frac{15}{8}\, \frac{\nu_f H}{S^2} \left(\sqrt{1 + \frac{\psi M}{30\,\mathrm{Pr}}} - 1 \right), \tag{12a}$$

or

$$\frac{\mathrm{Pe}^*}{\mathrm{Pr}} = 5 \left(\sqrt{1 + \frac{\psi M}{30\,\mathrm{Pr}}} - 1 \right). \tag{12b}$$

Since the flow is stabilized, $v = 0$. This follows, in particular, from (8), (3), and (4).

Equations (12a, b) include the parameter ψ, which depends on T_f. We approximate the fluid temperature profile in any cross section x by a polynomial of degree four, which should contain no odd powers of $\bar{y}$, by virtue of symmetry:

$$T_f = T_{ax} + a_2\bar{y}^2 + a_4\bar{y}^4. \tag{13}$$

Letting $v = 0$ in (2) and noting that $u = 0$ when $\bar{y} = 1$ so that, consequently, $\partial^2 T_f/\partial \bar{y}^2 = 0$, we obtain

$$a_4 = -\frac{1}{6}\, a_2.$$

The constant a_2 is determined from (5). Then

$$T_f = T_{ax} + \frac{5}{6}(T_w - T_{ax})\left(\bar{y}^2 - \frac{1}{6}\bar{y}^4\right). \tag{14}$$

The mean-flow-rate temperature of the fluid in the channel is

$$T_m = \frac{\int_0^1 uT_f d\bar{y}}{\int_0^1 ud\bar{y}} = T_{ax} + \frac{39}{175}(T_w - T_{ax}). \tag{15}$$

It follows from (15) that

$$T_{ax} = \left(T_m - \frac{39}{175}T_w\right) : \left(1 - \frac{39}{175}\right). \tag{16}$$

We find the relationship $T_m = T_m(x)$ from (2). We multiply both sides of the equation by $d\bar{y}/u_a$, and integrate with respect to $\bar{y}$ in the interval [0, 1]. Since u does not depend on x, while v = 0, after integration with allowance for T_m in (15), we obtain

$$\frac{1}{H}\frac{dT_m}{d\bar{x}} = \left[\frac{a_f}{S^2 u_a}\frac{\partial T_f}{\partial \bar{y}}\right]_{\bar{y}=1}. \tag{17}$$

From (14) and (16), we have

$$\left.\frac{\partial T_f}{\partial \bar{y}}\right|_{\bar{y}=1} = \frac{8}{5}(T_w - T_{ax}) = \frac{35}{17}(T_w - T_m) \tag{18}$$

We can then rewrite (17) as

$$\frac{1}{\Omega}\frac{dT_m}{d\bar{x}} + T_m = T_w; \quad \Omega = \frac{140}{17}\frac{1}{\mathrm{Pe}^*}. \tag{19}$$

The solution for this equation under the condition (7) has been given in [1]:

$$T_m(\bar{x}) = \exp(-\Omega\bar{x})\left[\Omega\int_0^{\bar{x}} T_w(\bar{x})\exp(\Omega\bar{x})\,d\bar{x} + T_0\right]. \tag{20}$$

We find the local heat-exchange coefficient as was done in [1]:

$$q = \alpha_x(T_w - T_m). \tag{21a}$$

On the basis of the Fourier law, the specific flux is

$$q = \lambda_f \left.\frac{\partial T_f}{\partial y}\right|_{y=S} = \frac{\lambda_f}{S}\frac{35}{17}(T_w - T_m). \tag{21b}$$

Comparing (21a) and (21b), we find

$$\mathrm{Nu}_x = \frac{70}{17} \approx 4.12. \tag{22}$$

The approximate expression (22) was obtained for any function $T_w(x)$ under the requirement that the indicated restrictions are satisfied; it is only strictly valid, however, for a stabilized fluid flow with constant heat-loss density. For isothermal walls, with stabilized flow $\mathrm{Nu}_x = 3.77$ [1]. As a consequence, for T_w = const and q = const, Eq. (22) gives relative errors of 9 and 0%, respectively.

The average heat-exchange coefficient, referred to the inlet temperature, is found in [1-3, 5, 6]

$$Q = \bar{\alpha}(\bar{T}_w - T_0)HB. \tag{23a}$$

We can write the expression for Q in a different form:

$$Q = BH\int_0^1\int_0^1 q d\bar{x}d\bar{z} = \alpha_x BH(\bar{T}_w - \bar{T}_m); \tag{23b}$$

$$\bar{T}_m = \int_0^1 T_m(\bar{x})\,d\bar{x}. \tag{23c}$$

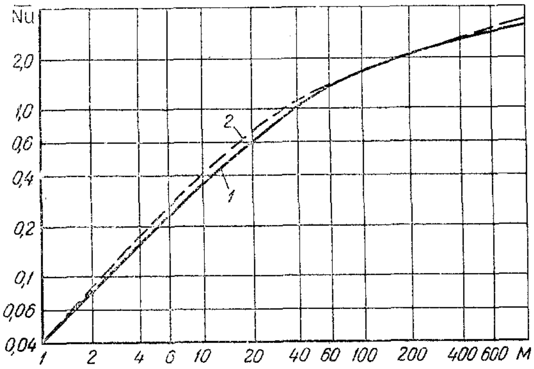

Fig. 2. Average Nusselt number as function of M = GrPrb/H: 1) from Eqs. (28), (29); 2) from [6].

Comparing (23a) and (23b), we find

$$\overline{\mathrm{Nu}} = \mathrm{Nu}_x \varkappa; \ \varkappa = \frac{\overline{T}_w - \overline{T}_m}{\overline{T}_w - T_0}. \tag{24}$$

We now find the relationship between ψ and $\varkappa$. We average (16) over x, and (14) over the channel volume; we then divide one of the resulting equations by the other and, allowing for the definitions of ψ and $\varkappa$, we obtain

$$\psi = 1 - \frac{14}{17}\varkappa. \tag{25}$$

From (12b) and (20), (24), (25) we can determine Nu_x and $\overline{\mathrm{Nu}}$ as functions of the channel geometry, the physical properties of the fluid, and the wall temperature.

Special case: the channel walls are isothermal,

$$T_w(\bar{x}) = \mathrm{const} = \Theta. \tag{26}$$

From (23) we have

$$T_m = \Theta - (\Theta - T_0)\exp(-\Omega\bar{x}), \tag{27}$$

and, consequently,

$$\varkappa = \frac{1 - \exp(-\Omega)}{\Omega}; \ \overline{\mathrm{Nu}} = \mathrm{Nu}_x \frac{1 - \exp(-\Omega)}{\Omega}. \tag{28}$$

Solving (12b) for M, with allowance for (19) and (28) we obtain

$$M = \frac{120}{\mathrm{Pr}\,\Omega^2}\frac{14}{17}\frac{\left(\frac{14}{17} + \Omega\,\mathrm{Pr}\right)}{1 - \frac{14}{17}\frac{1 - \exp(-\Omega)}{\Omega}}. \tag{29}$$

Equations (29) and (28), which contain the common parameter Ω, enable us to find the relationship $\overline{\mathrm{Nu}} = \overline{\mathrm{Nu}}(M)$, shown in Fig. 2 (solid line) and Fig. 3 (curve a) for air (Pr = 0.72).

With Pr = 0.72, we have the following relationship for $M \le 0.43$, with a relative error not exceeding 1%:

$$M = \frac{120}{\Omega}\frac{14}{17}; \ \overline{\mathrm{Nu}} = \frac{70}{17}\frac{1}{\Omega}$$

while the asymptotic expression (30) is valid

$$\overline{\mathrm{Nu}} = \frac{1}{24}M. \tag{30}$$

Let us find the limits of applicability of (28) and (29) in the region of large M. It has been shown in [1] for T_W = const that the conditions for temperature-profile stabilization are satisfied with an error of less than 1% if

$$\frac{l_{it}}{b} \geqslant 0.055\,\mathrm{Pe}. \tag{31}$$

Letting $\overline{L}_{it} = l_{it}/H$, we rewrite (31) as

$$\overline{L}_{it} \geqslant 0.055\,\mathrm{Pe}^*.$$

If, for example, we require that $\overline{L}_{it} \le 0.1$, then $\mathrm{Pe}^* \le 1.82$. Solving (29) and (12b) simultaneously for $\mathrm{Pe}^* = 1.82$, we obtain $M \le 33.2$ for $\overline{L}_{it} \le 0.1$. Since for gases ($\mathrm{Pr} \approx 1$), the lengths of the initial thermal and hydrodynamic sections are similar, the inequalities obtained characterize the strictness of the assumptions (8).

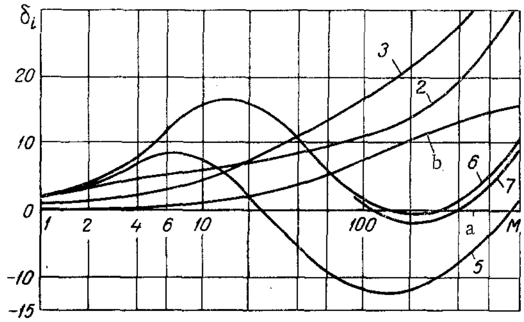

Fig. 3. Comparison of calculated and experimental values $\overline{\mathrm{Nu}} = \overline{\mathrm{Nu}}(M)$: 2) according to [2]; 3) [3]; 5) [5]; 6) [6]; 7) [7]; a) from formulas (28) and (29); b) from formulas (28) and (33).

In the indicated range of variation in M, we would expect about a 10% error in determination of $\overline{\mathrm{Nu}}$. In actuality, we can use the relationships for Nu_x and $\overline{\mathrm{Nu}}$ over a wider range while keeping this accuracy. As an example, in Fig. 2 we show $\overline{\mathrm{Nu}} = \overline{\mathrm{Nu}}(M)$ for Pr = 0.72 according to [6] (dashed line).

We note that if in approximating the temperature field we use a polynomial of degree two, we obtain the expression

$$\mathrm{Nu}_x = 5 \tag{32}$$

for the local Nusselt number. In this case we can find $\overline{\mathrm{Nu}} = \overline{\mathrm{Nu}}(M)$ (curve b, Fig. 3) by solving (28) and (33) simultaneously, while making allowance for (32):

$$M = \frac{120}{\mathrm{Pr}\,\Omega^2}\,\frac{1+\Omega\,\mathrm{Pr}}{1-\dfrac{5}{6}\,\dfrac{1-\exp(-\Omega)}{\Omega}}\,. \tag{33}$$

The results obtained by various investigators are compared in Table 1 and Fig. 3. The figure gives curves for $\delta_{\mathrm{Nu}} = f(M)$. The relative deviations in the average Nusselt numbers (δ_{Nu}) are found from the formula

$$\delta_{\mathrm{Nu}} = \frac{\overline{\mathrm{Nu}}_i - \overline{\mathrm{Nu}}_0}{\overline{\mathrm{Nu}}_0}\ 100\,\%.$$

Here i indicates the numbered reference; $\overline{\mathrm{Nu}}_0$ is found from (28) and (29). We represent the results in this form since in $\overline{\mathrm{Nu}} = \overline{\mathrm{Nu}}(M)$ coordinates, all curves are nearly the same (Fig. 2).

Convection has been investigated in [2] for a flat channel, with allowance for the initial section, where the temperature profiles are not established. The data are obtained on the assumption that the fluid at the inlet is at the ambient temperature, while the velocity profile is a plane. This last assumption is not valid and, obviously, determines the accuracy of the results. The equations of the plane boundary layer are given in criterial form; they are solved numerically by a finite-difference method for Pr = 0.7. For small M, the authors of [2] assume the relationship (30), and for large M ($M > 10^3$), the asymptotic relationships

$$\overline{\mathrm{Nu}} = 0.68\ M^{1/4}\,. \tag{34}$$

Multiplying both sides of (34) by H/b, we find

$$\overline{\mathrm{Nu}}_H = 0.68\,(\mathrm{Gr}_H \mathrm{Pr})^{1/4}\,, \tag{35}$$

which characterizes heat exchange in a single layer of height H in an unbounded medium. The results of [2] are compared with the data of other investigators in Fig. 3 (curve 2).

Free convection in a vertical slot with isothermal walls has been considered in [3]. The author recommends the following computational formulas for determining $\overline{\mathrm{Nu}}_S = \mathrm{Nu}_S(M_S)$:

$$\overline{\mathrm{Nu}}_S = \frac{1-\dfrac{8}{\pi^2}\exp\left(-\dfrac{\pi^2}{4}\,\sigma\right)}{\sigma}\,;\quad \sigma = (\mathrm{Pe}_S^*)^{-1}; \tag{36a}$$

$$M_S = \frac{3\,\mathrm{Pr} + 0.75}{\mathrm{Pr}\,\sigma^2\left[1-\dfrac{1}{3\sigma}+\dfrac{32}{\pi^4\sigma}\exp\left(-\dfrac{\pi^2}{4}\,\sigma\right)\right]}\,. \tag{36b}$$

This relationship is given graphically in [3]. Figure 3 (curve 3) gives a comparison with the data of other investigators.

When M < 1, the asymptotic relationship (30) is valid with an error of less than 1%. When M > 320, heat exchange in the channel is almost the same as that for a single layer. Here (36a, b) yield

$$\overline{\mathrm{Nu}} = 0.65\ M^{1/4}\,. \tag{37}$$

In an experimental study of natural convective heat exchange in a flat vertical channel with isothermal walls [5] the following empirical formula was obtained:

TABLE 1. Comparison of Formulas for Average Nusselt Number in Channels for Large M with Data for Single Layer

c	n	Source	Heat-exchange condition
0,68	0,25	[2]	T_w=const
0,65	0,25	[3]	"
4,12	0	Present study	"
0,566	0,25	[5]	"
0,600	0,25	[6]	"
0,600	0,25	[7]	q=const
0,585	0,25	[8]	T_w=const
0,55	0,25	[4]	"
0,54	0,25	[9]	"
0,525	0,25	[10]	"
0,52	0,25	[11]	"
0,625	0,25	[12]	q=const

$$\overline{\mathrm{Nu}} = \frac{1}{24} M \left\{1 - \exp\left[-\left(\frac{32.4}{M}\right)^{3/4}\right]\right\}. \tag{38}$$

When M < 4, the asymptotic representation (30) is correct, with an error of less than 1%, while when M > 1500, the equation

$$\overline{\mathrm{Nu}} = 0.556\, M^{1/4}$$

is correct to within 5%. Figure 3 illustrates Eq. (38) (curve 5). In [6], the experiments were carried out with square plates, so that there could be lateral leakage of heat by conduction, particularly for small M. The empirical formula is

$$\overline{\mathrm{Nu}} = \frac{1}{24} M \left[1 - \exp\left(-\frac{35.1}{M}\right)\right]^{3/4}. \tag{39}$$

When M < 7, Eq. (30) holds to within 1%, while for M > 1750, the asymptotic form

$$\overline{\mathrm{Nu}} = 0.6\, M^{1/4} \tag{40}$$

also holds to within 1%. Figure 3 (curve 6) illustrates Eq. (39).

An experimental investigation of the natural convection of air flows in straight and stepped short channels has been carried out for constant heat-loss density at the walls [7]. The experimental results were processed in criterial form. The computational formulas given in [7] for the straight channel can be represented in the following form, after uncomplicated manipulations:

$$\overline{\mathrm{Nu}} = 0.6\,(M)^{1/4}\,;\ M > 87. \tag{41}$$

We note that the average Nusselt number for stepped channels can exceed $\overline{\mathrm{Nu}}$ for straight channels by 38%. Figure 3 (curve 7) illustrates Eq. (41).

Since when $M > 10^3$, the channel conditions differ little from the heat-exchange conditions for a single layer (plate), it is interesting to compare the asymptotic representations of $\overline{\mathrm{Nu}} = \overline{\mathrm{Nu}}(M)$ with the criterial relationship for a single layer:

$$\overline{\mathrm{Nu}}_H = c\,(\mathrm{Gr}_H \mathrm{Pr})^n.$$

Table 1 gives the values of c and n obtained by various authors, in systematic form.

Analysis of the curves of Fig. 3 shows that the data of the various authors for $\overline{\mathrm{Nu}}$ can differ by 30% when $0 \le M \le 10^3$. The differences in the analytic relationships (curves a, b, 3, 5) result basically from the choice of model; they are affected to a smaller extent by the rigor of the mathematical solution. The spread in the experimental data is characteristic, as is the tendency to be too high as compared with most of the results for single layers (Table 1).

The asymptotic expressions for Nu and for large M, calculated from (28), (29), are physically meaningless, but the authors do not recommend that the results be used for M > 1000.

For practical determinations of $\overline{\mathrm{Nu}}$, it is most convenient to use the empirical relationships [5, 6] based on (38), (39). Since our results are in satisfactory agreement with the data of other investigators, we can recommend our relationships for the analysis of local heat exchange and the study of temperature fields in channel walls.

NOTATION

T_w is the local temperature of the channel wall;
T_f is the local temperature of the fluid in the channel;
T_0 is the temperature of the fluid ahead of the channel inlet;
T_{ax} is the temperature on the channel axis;
$\overline{T}_w$ is the average wall temperature;
T_{fv} is the mean-volume temperature of the fluid;
T_m is the mean-flow temperature of the fluid in cross-section x;

$\overline{T}_m$	is the average value of T_m over the channel height;
H	is the channel height;
b = 2S	is the channel width;
B	is the depth of the channel;
$\overline{x}$, $\overline{y}$, $\overline{z}$	are the relative coordinates;
u, v	are the longitudinal and transverse velocity components;
u_{ax}	is the velocity on the channel axis;
u_a	is the average velocity over a channel cross section;
ν_f, μ_f	are the kinematic and dynamic viscosities of the fluids;
$\beta = 1/T_0$	is the coefficient of thermal expansion of the fluid;
g	is the free-fall acceleration;
ρ_f, a_f, λ_f	are the density, thermal diffusivity, and thermal conductivity of the fluid;
R	is the friction force;
A	is the Archimedes force;
ΔI	is the change in momentum;
Gr	is the Grashof number;
Pr	is the Prandtl number;
Pe	is the Peclet number;
α_x, $\overline{\alpha}$	are the local and mean heat-exchange coefficients;
Nu_x, $\overline{Nu}$	are the local and average Nusselt numbers;
Q	is the total heat flux dissipated by the channel wall;
Θ	is the temperature of the isothermal wall;
q	is the density of the heat flux dissipated by the wall;
$\overline{Nu}_S$, Gr_S, Pe_S	are the Nusselt, Grashof, and Peclet numbers for the controlling dimension S.

LITERATURE CITED

1. B. S. Petukhov, Heat Exchange and Resistance in Laminar Flow of Fluid Through Heights [in Russian], Énergiya (1967).
2. J. R. Bodoia and J. F. Osterl, Trans. ASME, Ser. C, 84, No. 1 (1962).
3. L. T. Fuks, Izv. Vys. Uch. Zav. Énergetika, No. 3 (1961).
4. E. R. Eckert and R. M. Drake, Heat and Mass Transfer, McGraw–Hill, New York (1959).
5. Toshio Aihara, Trans. Japan Soc. Mechan. Engrs., 29, No. 201 (1963).
6. W. Elenbaas, Physica Holland, 9, No. 1 (1942).
7. N. Sobel, F. Landis, and W. K. Mueller, Proc. Third International Heat Transfer Conference, Chicago, Illinois, US, ASME, No. 53 (August 1963).
8. S. Ostrah, Naca TR 1111 (1953).
9. M. A. Mikheev, Fundamentals of Heat Transfer [in Russian], GEI (1957).
10. E. Shmidt and W. Becman, Ing. Wes., 1 (1930).
11. G. Greber, S. Erk, and U. Grigull, Fundamentals of Heat Exchange [Russian translation], IL (1958).
12. E. H. Sparrow and J. L. Gregg, Trans. ASME, 78, No. 2, 435)440 (1956).